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Probability & Statistics,

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Normal Distribution

A continuous random variable X is said to have normal distribution with parameters μ and σ if its probability density function is given by

$$f(x) = \frac{1}{(\sqrt{2\pi})\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

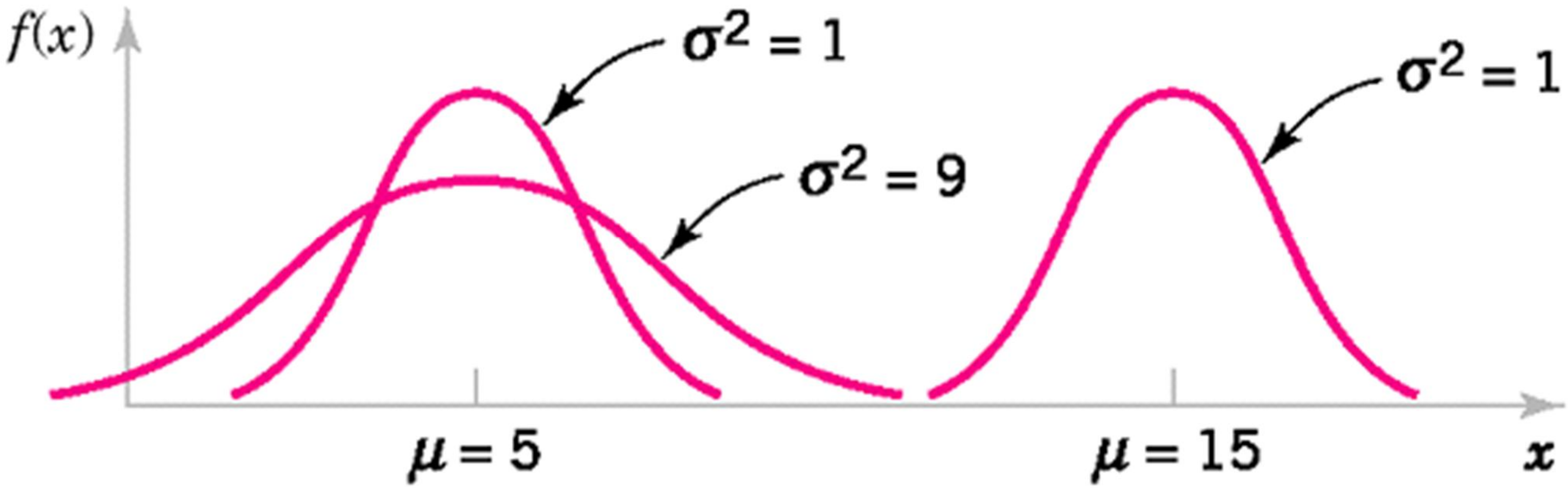
$$\text{where } -\infty < \mu < \infty \text{ and } \sigma > 0.$$

Normal Distribution

The normal distribution is also known as Gaussian distribution because the Gaussian function which is defined as

$$f(x) = ae^{-(x-b)^2/c^2}$$

Normal Distribution



The parameter μ is called shape parameter and σ^2 is called scale parameter

Normal Distribution

Mean of Normal distribution:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x}{\sigma} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma} + \frac{\mu}{\sigma} \right) e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma\, u e^{-u^2/2}\,du + \mu \int_{-\infty}^{\infty} f(x)\,dx.$$

$$= 0 + \mu = \mu.$$

Normal Distribution

Variance of Normal distribution:

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2/2} du, \text{ put } u = \frac{x-\mu}{\sigma}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} u^2 e^{-u^2/2} du$$

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Normal Distribution

Variance of Normal distribution:

$$Var(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty u^2 e^{-u^2/2} du$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left(\left[u.\int ue^{-u^2/2} du \right]_0^\infty - \int_0^\infty 1.\left(\int ue^{-u^2/2} du \right) du \right)$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \left(\left[-ue^{-u^2/2} \right]_0^\infty + \int_0^\infty e^{-u^2/2} du \right)$$

$$= 0 + \sigma^2.$$

Normal Distribution

Moment generating function of Normal distribution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 tx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 t(x-\mu) + \sigma^4 t^2 - \sigma^4 t^2 - 2\sigma^2 \mu t\right]} dx$$

Normal Distribution

Moment generating function of Normal distribution(cont.):

$$= e^{\frac{\sigma^2 t^2}{2}+\mu t} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2-2\sigma^2 t(x-\mu)+\sigma^4 t^2\right]} dx$$

$$= e^{\frac{\sigma^2 t^2}{2}+\mu t} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)-\sigma^2 t\right]^2} dx$$

$$= e^{\frac{\sigma^2 t^2}{2}+\mu t}$$

Normal Distribution

Mean, variance and moment generating function of the Normal distribution:

$$E(X) = \mu$$

$$Var(X) = \sigma^2$$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Standard Normal Distribution

Normal Distribution (Special Case):

A normal random variable with mean 0 and variance 1 is called standard normal random variable.

Note: If X is normal with mean μ and variance σ^2, then $\dfrac{X-\mu}{\sigma}$ is standard normal distribution.

Standard Normal Distribution

If Z is a standard normal random variable, then the density function of Z is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

Advantage of standard normal distribution

Remark:

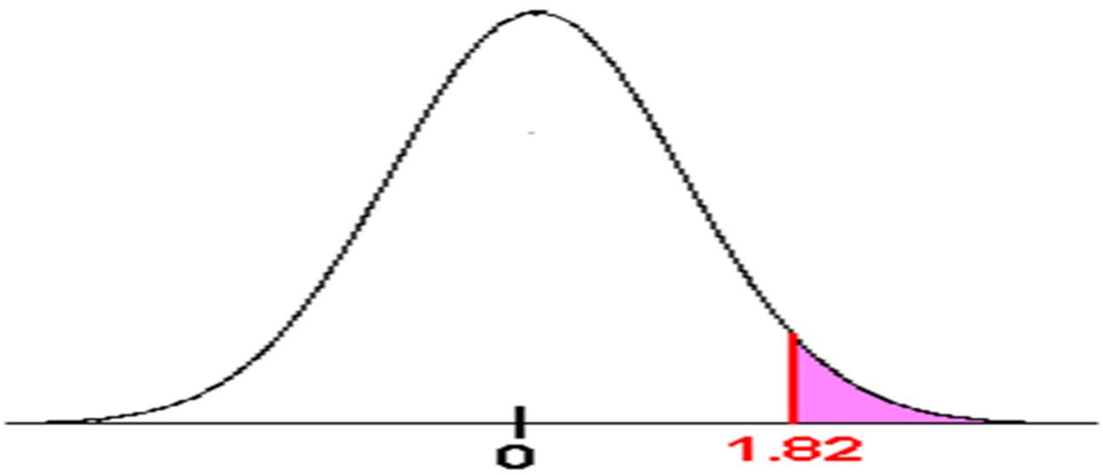
- Cumulative distribution values available.

- Every distribution (discrete/continuous) can be approximated by normal distribution and hence standard normal distribution (the proof required central limit theorem which will be discussed in Chapter 7).

Standard normal distribution

Notation:

$$P(Z \geq z_{\alpha}) = \alpha$$



$$z_{0.0344} = 1.82 \text{ since } P(Z > 1.82) = 0.0344.$$

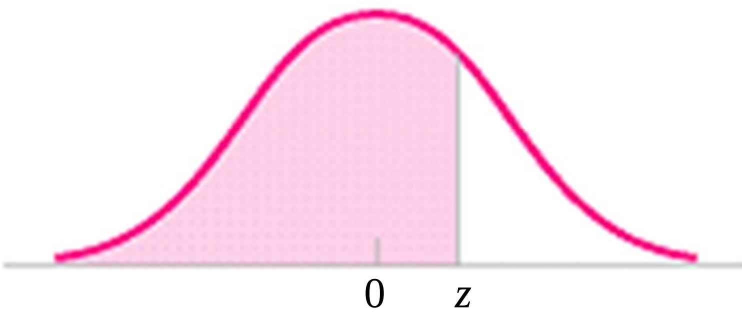
Standard normal distribution

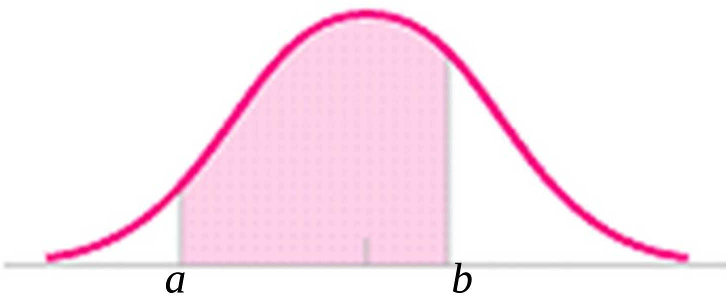
How to use cumulative distribution Table:

TABLE V
Cumulative distribution: Standard normal (*concluded*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.523[illegible]
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.563[illegible]
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.602[illegible]
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.64[illegible]
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.67[illegible]
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7[illegible]
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7[illegible]
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7[illegible]
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.[illegible]
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.[illegible]
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	[illegible]
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	[illegible]
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	[illegible]
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	0.9236	0.9251	0.9265	[illegible]

Standard normal Distribution





The standard normal probabilities
$F(z) = P(Z \leq z)$

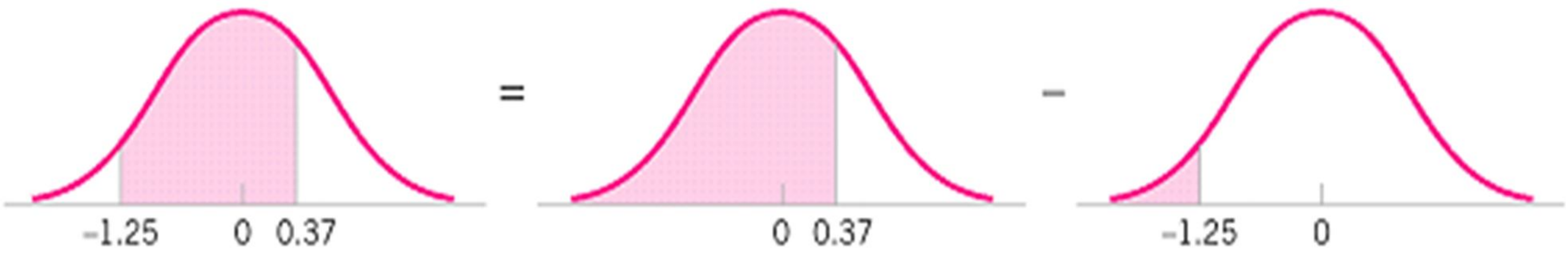
The standard normal probability
$F(b) - F(a) = P(a \leq Z \leq b)$

Problem 1: $F(-z) = 1 - F(z)$ (verify yourself)

Standard normal distribution

Problem 2: **Find the probabilities that a random variable having the standard normal distribution will take on a value (a) between –1.25 and 0.37; (b) greater than 1.26; (c) greater than –1.37.**

(a) $P(-1.25 < Z < 0.37) = F(0.37) - F(-1.25) = 0.6443 - 0.1056$
$= 0.5387$



Standard normal distribution

Problem 3: Find (a) $z_{0.01}$; (b) $z_{0.05}$

Since $F(z_{0.01}) = 1 - 0.01 = 0.99$, we look for the entry in Table which is closest to 0.99 and get 0.9901 corresponding to $z = 2.33$. Thus $z_{0.01} = 2.33$.

Similarly try part (b)

Normal to standard normal

If a random variable *X* has a normal distribution with the mean μ and the standard deviation σ, then, the probability that the random variable *X* will take on a value less than or equal to *a*, is given by

$$P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right) = P\left(Z \le \frac{a-\mu}{\sigma}\right) = F\left(\frac{a-\mu}{\sigma}\right)$$

Similarly

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

Standard normal distribution

Problem 4:

If a random variable has the normal distribution with $\mu = 16.2$ and $\sigma^2 = 1.5625$, find the probabilities that it will take on a value (a) greater than 16.8;

(b) less than 14.9; (c) between 13.6 and 18.8.

Solution:

$$(a)\ P(X > 16.8) = 1 - P(X \le 16.8) = 1 - F\left(\frac{16.8 - 16.2}{1.25}\right)$$

$$= 1 - F(0.48) = 1 - 0.6844 = 0.3156 .$$

$$(b)\ P(X < 14.9) = F\left(\frac{14.9 - 16.2}{1.25}\right) = F(-1.04) = 0.1492.$$

Problem 5

The average life of bulb is 1000 hours and the standard deviation is 300 hours. If X is the life period of a bulb which is distributed normally, find the probability that a randomly picked bulb will function for less than 500 hours.

Problem 6

Suppose that the load of an airplane wing is a random variable X with N(1000; 14400) distribution. The maximum load that the wing can withstand is a random variable Y which is N(1260; 2500). If X and Y are independent, find the probability that the load encountered by the wing is less than its critical load.

Problem 7

Let Z be a Standard Normal random variable. Show that Z^2 follows Chi-Squared with degree of freedom 1.

Problem 8

Suppose scores of 300 students in Statistics course has follows normal distribution. If an instructor want to assign A grades to those scores above $\mu + \alpha$ then how many students will get A grades ?